GROUP REPRESENTATIONS AND ANALYSIS

by George W. Mackey

LECTURE I

In these three lectures I shall try to give the non-specialist some idea of what the theory of group representations is about, especially in its analytical aspects. This theory was begun in 1896 by Frobenius as a branch of abstract algebra. However, in the 1920's and 1930's it was extended so as to apply to continuous groups acting in infinite dimensional function spaces and recognized to be intimately related to Fourier analysis, expansions in spherical harmonics, and other analytical theories.

Let G be a group. By a *representation* of G one usually means a linear representation; that is, the system consisting of a vector space V and an assignment of a linear transformation L_x in V to each x in G in such a manner that the following identity holds:

$$L_{xy} = L_x L_y.$$

We shall assume throughout that V is a vector space over the complex numbers. Here is a typical example of how group representations arise in analysis. Let S be the surface of the unit sphere about the origin in three-space. Let G be the group of all rotations about the origin. For each x in G, and each s in S, let sx denote the result of transforming s by the rotation x. Then for each complex valued function f on S we may define the translate f_x of f by x as the function $s \to f(sx)$. Let V be any vector space of complex valued functions on S such that f_x is in V whenever f is in V and x is in G; V might be the space of all continuous functions on S for example. Then for each x, $f \to f_x$ is a linear transformation L_x and $x \to L_x$ is a representation of G.

Let L be any representation of a group G and let V_1 be any linear subspace of the space V of L. If $L_x(f)$ is in V_1 for all x in G and all f in V_1, we shall say that V_1 is an *invariant* subspace of V. If V_1 is invariant, then we obtain a new representation by restricting each L_x to V_1. We shall call this the *subrepresentation* defined by V_1 and denote it by L^{V_1}. A primary

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goal in the study of group representations is to show that the representations one encounters can be reconstructed in some more or less transparent way from subrepresentations having a simple structure. Here is an almost trivial example, which is nonetheless typical. Let S be the real line and let G be the cyclic group of order 2 consisting of the transformations $s \to s$ and $s \to -s$. Let V denote the vector space of all complex valued functions on S and let L denote the representation of G such that $L_x(f)(s) = f(sx)$. Let V^e denote the subspace of V consisting of all even functions and let V^0 denote the subspace consisting of all odd functions. Then V^e and V^0 define subrepresentations L^{V^0} and L^{V^e} which have a very simple structure. In fact each $L_x^{V^0}$ and each $L_x^{V^e}$ is simply multiplication by one or minus one. Since each f in V is uniquely the sum of a member of V^e and a member of V^0, it follows that L is *equivalent* to the *direct sum* of L^{V^0} and L^{V^e} in the sense of the following definitions. Given two representations L^1 and L^2 of the same group G, their direct sum $L^1 \oplus L^2$ is the representation such that $(L^1 \oplus L^2)_x(f_1, f_2) = L_x^1(f_1), L_x^2(f_2)$. Its space is the space of all pairs f_1, f_2 where f_1 is in the space of L^1 and f_2 is in the space of L^2. Two representations L^1 and L^2 are said to be *equivalent* if there exists a one-to-one linear transformation W from the space of L^1 onto the space of L^2 such that $WL_x^1 W^{-1} = L_x^2$ for all x in G.

The analysis we have just given of a particular representation of the cyclic group of order two can be easily extended so as to apply to any representation of any finite commutative group. Let us define a *character* of a finite group G to be a function χ from G to the complex numbers such that $\chi(xy) = \chi(x)\chi(y)$ for all x and y in G. The product of two characters is itself a character and under this operation the characters themselves form a group which we denote by $\hat{G}$ and call the *character group*. It is easy to see that $\sum_{x \in G} \chi(x) = 0$ for any character χ which is not identically one, and it follows that $\sum_{x \in G} \chi_1(x)\overline{\chi_2}(x) = 0$ for any two distinct characters χ_1 and χ_2. This "orthogonality relation" implies that the characters are linearly independent and hence that there cannot be more than $O(G)$ of them, where $O(G)$ is the number of elements in G. As a matter of fact it can be proved that $O(G) = O(\hat{G})$ and even that G and $\hat{G}$ are isomorphic. Now let L be any representation of G and let V be the space of the representation. For each f in V and each χ in $\hat{G}$ let us define f_χ as

$$(1/O(G)) \sum_{x \in G} \bar{\chi}(x) L_x(f).$$

We verify at once that $(f_\chi)_\chi = f_\chi$ and that f is of the form f_χ if and only if $L_x(f) = \chi(x)f$. Let V^χ denote the set of all f with either and hence both of these properties. It is obvious that V^χ is an invariant subspace of V

and that in each V^χ, L_x is simply multiplication by the constant $\chi(x)$. The key fact, easy but not quite trivial to prove, is that for all f

$$f = \sum_{\chi \in \hat{G}} f_\chi .$$

It follows from the orthogonality relation that $(f_{\chi_1})_{\chi_2} = 0$ if $\chi_1 \neq \chi_2$ and hence that the above decomposition is unique. Thus L is equivalent to the direct sum of the L^{V^χ}. In the special case in which G is of order 2 there are just two characters and our decomposition is a slight generalization of that obtained from even and odd functions.

This decomposition may be reformulated so as to apply to representations of any finite group, commutative or not, but only if we admit somewhat more complicated direct summands. Non-commutative groups always admit representations which are *irreducible* in the sense of having no proper subrepresentations and yet have spaces which are not one-dimensional. In order to describe the nature of the decomposition which is possible, it will be convenient to introduce some more definitions. Let L^1 and L^2 be representations of the group G acting in spaces V^1 and V^2. If V^3 is an invariant subspace of V^1, and V^4 is an invariant subspace of V^2, then the smallest linear subspace, $V^3 \dotplus V^4$, which contains V^3 and V^4 will be an invariant subspace of the space $V^1 \oplus V^2$ of $L^1 \oplus L^2$. It may happen that *every* invariant subspace of $V^1 \oplus V^2$ is of this form. If so, we shall say that L^1 and L^2 are *disjoint*. Let V^1 be an invariant subspace of the space V of a representation L. Suppose that there exists a second invariant subspace such that $V^1 \cap V^2 = 0$ and $V^1 \dotplus V^2 = V$, so that L is equivalent to $L^{V_1} \oplus L^{V_2}$. It can be shown that if V^2 exists and is such that L^{V_1} and L^{V_2} are disjoint, then V^2 is unique. In this case we shall say that V^1 (and V^2) are *central* invariant subspaces and that V^2 is the *complement* of V^1. It can be shown that $V^1 \dotplus V^3$ and $V^1 \cap V^3$ are both central wherever V^1 and V^3 are central. More generally, if $V^1, V^2, \cdots, V^k$ are all central invariant subspaces, then there exists a direct sum decomposition $L \simeq L^{W_1} \oplus \cdots \oplus L^{W_n}$ such that each W_j is central and each V^j is the linear union of some of the W_j. It follows that the central invariant subspaces form a Boolean algebra. We shall call this the *center* of the representation. If the center is trivial, that is, if there are no proper central invariant subspaces, we shall say that the representation is *primary*. If the center is finite, then each minimal element defines a primary subrepresentation and we have our representation decomposed as a direct sum of mutually disjoint primary subrepresentations. We shall call this decomposition the *canonical primary decomposition*.

It is easy to see that the decomposition $L = \sum_{\chi \in \hat{G}} \oplus L^{V^\chi}$ defined above for representations of finite commutative groups is just the canonical primary decomposition. It can be shown that every representation of every finite group has a finite center and hence a canonical primary decomposition. We wish to define the notion of character for finite non-commutative groups in such a manner that we may obtain this decomposition by formulas analogous to those used in the commutative case. Given any finite dimensional representation L of a group G, we may define a complex valued function χ^L on G by setting $\chi^L(x) = \text{Trace}(L_x)$. It is easy to see that $\chi^L(sxs^{-1}) = \chi^L(x)$ so that χ^L is constant on the conjugate classes. Also $\text{Trace}(WL_xW^{-1}) = \text{Trace}(L_x)$; so $\chi^L(x) \equiv \chi^{L^1}(x)$ whenever L and L^1 are equivalent. Less trivial is the important fact that $\chi^L(x) \equiv \chi^{L'}(x)$ implies that L and L' are equivalent representations. When L acts in a one-dimensional vector space, it is necessarily of the form $x \to \chi(x)I$, where χ is a character, and for such a representation $\chi^L(x) \equiv \chi(x)$. Thus the characters of a finite commutative group are just the functions χ^L where L is an irreducible representation. It is customary to call the function χ^L the *character* of the representation L and to call the character of an irreducible representation an *irreducible character*. For consistency's sake what we previously called the characters of a finite commutative group should be called its *irreducible characters*. In any event one has irreducible characters for any group with finite dimensional irreducible representations. Using them, one can formulate the desired decomposition theorem almost exactly as before. Given an arbitrary representation L of an arbitrary finite group G, let χ be any irreducible character of G. For each f in the space V of L let $f_\chi = [1/O(G)] \sum_{x \in G} \overline{\chi(x)} L_x(f)$. Then $L_y(f_\chi) = (L_y(f))_\chi$ so that the set of all members of V of the form f_χ is an invariant subspace. We denote it by V^χ. There are only a finite number of χ's and we have $f = \sum_{\chi \in \hat{G}} f_\chi$ where $\hat{G}$ is the set of all irreducible characters of G. (In the non-commutative case, $\hat{G}$ is not a group—the product of two irreducible characters is not an irreducible character.) One has the orthogonality relations

$$\sum_{x \in G} \chi_1(x) \overline{\chi_2(x)} = 0 \quad \text{if } \chi_1 \neq \chi_L$$

from which it follows that $[f_{\chi_1}]_{\chi_2} = 0$ and that L is a direct sum of the L^{V^χ}. This decomposition is easily shown to be the canonical primary one. Each L^{V^χ} has the property that every irreducible subrepresentation has χ as its character and hence is equivalent to every other. Actually, one can find (in many ways) a family $\{V^\alpha\}$ of irreducible subspaces, such that each f is uniquely a finite sum, $f = f_{\alpha_1} + f_{\alpha_2} + \cdots + f_{\alpha_n}$, where $f_{\alpha_j} \in V^{\alpha_j}$. In this sense every primary representation of a finite group is a direct sum of (possibly infinitely many) equivalent irreducible representations. This decomposition is not unique, but the equivalence class of the irreducible components is, and so is the multiplicity. Thus the most general representation of a finite group G is described to within equivalence by a cardinal number valued function on the finite set $\hat{G}$.

Because of the theory just described, one knows all representations of a finite group G when one knows the irreducible representations of G. Moreover, one knows that, to within equivalence, each finite group G has just a finite number of distinct irreducible representations. The problem of finding these and their characters is one of the main problems of the theory. It has been solved for many important groups D but remains unsolved for others. We conclude this lecture with a few general facts and an example.

Given any finite group G, one can define a representation (called the regular representation) as follows. V is the vector space of all complex valued functions on G and for each x in G, L_x is the linear transformation which takes the function $y \to f(y)$ into the function $y \to f(yx)$. This representation has the property that its decomposition into irreducible subrepresentations contains every irreducible representation of G and contains it a number of times equal to the dimension of the space in which it acts. Since the dimension of V is $O(G)$, it follows that the sum of the squares of the dimensions of the irreducible representations of G is equal to $O(G)$. For example, let G be the non-commutative group of order 6, that is, the group of all permutations of three elements. Since $3^2 > 6$ there can exist no irreducible representation whose dimension is greater than 2. Since $2^2 + 2^2 > 6$, there can exist at most one whose dimension is 2. Since only commutative groups have all one-dimensional irreducible representations, we must have exactly one two-dimensional irreducible representation. Since $6 - 2^2 = 2$ there are two one-dimensional irreducible representations. It is easy to determine the characters of all three irreducible representations using the orthogonality relation, the fact that $\chi^L(e) =$ dimension of L, the fact that every character is a constant on the conjugate classes, and the fact that a one-dimensional character must be 1 on the commutator subgroup. Here is the result:

	e	(abc)	(bac)	(ab)	(ac)	(bc)
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1
χ_3	2	-1	-1	0	0	0

Note that the number of equivalent irreducible representations is equal to the number of classes of conjugate elements. This is a general theorem. In fact, the irreducible characters constitute a basis for the vector space of all functions which are constant on the conjugate classes.

LECTURE II

In this and the final lecture, I shall discuss the extent to which the theory of the first lecture may be extended so as to apply to infinite groups, both continuous and discrete. By way of motivation for this extension, I shall begin by exhibiting a remarkable analogy between a special case of the character decomposition theorem and the formulae relating a periodic function to its Fourier coefficients. Let G be a finite commutative group and let L be its regular representation. Then V is the set of all complex valued functions on G and the formula $f_\chi = [1/O(G)] \sum_{x \in G} \bar{\chi}(x) L_x(f)$ becomes $f_\chi(y) = [1/O(G)] \sum_{x \in G} \bar{\chi}(x) f(yx)$. But $\sum_{x \in G} \bar{\chi}(x) f(yx) = \sum_{x \in G} \bar{\chi}(y^{-1}x) f(yy^{-1}x) = \sum_{x \in G} \chi(y) \overline{\chi(x)} f(x)$. Thus, f_χ is a constant $c(\chi)$ times the character χ where $c(\chi) = [1/O(G)] \Sigma f(x) \bar{\chi}(x) dx$. Since $f = \sum_{\chi \in \hat{G}} f_\chi$, we have the following theorem: Every function f on G has an expansion of the form $f(x) = \sum_{\chi \in \hat{G}} c(\chi)\chi(x)$ where $c(\chi) = [1/O(G)] \sum_{x \in G} f(x) \bar{\chi}(x)$. Let us compare these formulae with those of the L^2 theory of Fourier series: Every complex valued function f on the real line which has period 2π and is in L^2 on every interval of length 2π has an expansion of the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where convergence is in the mean and $c_n = (1/2\pi) \int_0^{2\pi} f(x) e^{-inx} dx$. To emphasize the analogy, let us write $c_n = c(n)$, $e^{inx} = \chi_n(x)$ and $e^{-inx} = e^{\overline{inx}} = \overline{\chi_n(x)}$. The formulae then become $f(x) = \sum_{n=-\infty}^{\infty} c(n)\chi_n(x)$, $c(n) = (1/2\pi) \int_0^{2\pi} f(x) \overline{\chi_n(x)}\, dx$. That the analogy is quite complete becomes clear when we realize that a function of period 2π is essentially a function on the group obtained from the additive group of the real line by identifying points which differ by an integral multiple of 2π. The functions e^{inx} are precisely the (continuous) characters of this group and $(1/2\pi) \int_0^{2\pi} f(x) \bar{\chi}_n(x) dx$ is just the average of $f(x)\bar{\chi}_n(x)$ over the group. We shall find that the Fourier expansion theorem is a corollary of a general theorem about group representations.

Henceforth, we shall deal only with topological groups which are *separable* in the sense of having a countable basis for the open sets and are *locally compact* in the sense that every point has a neighborhood whose closure is compact. We recall that a topological space is said to be compact if every covering by open sets has a finite subcovering. Any group is locally compact in the topology in which every set is open and such a group is separable when it has at most countably many elements. Thus finite groups and countable discrete groups are included. Here are some further examples of separable locally compact groups. a) The additive group of a finite dimensional vector space. b) The group of all non-singular $n \times n$ complex matrices. c) The group of all non-singular $n \times n$ real matrices. d) The group of all non-singular $n \times n$ matrices with p-adic numbers as entries. e) Any closed subgroup of any of the foregoing. The importance of the hypothesis of local compactness is due to a fundamental theorem of Haar about the existence of invariant measures in such groups. If G is separable and locally compact, then there exists a measure μ on G defined on all Borel sets, finite on compact sets, and not identically zero such that $\mu(Ex) = \mu(E)$ for all Borel sets E and all x in G. This measure (called right invariant Haar measure) is unique up to multiplication by a positive constant. When G is the additive group of a finite dimensional vector space the Haar measure may be taken to be Lebesque measure. When G is a countable discrete group, the Haar measure of a set E may be taken to be the number of points in E. In this last case $\int_G f(x)\,du(x) = \sum_{x \in G} f(x)$. More generally, integration with respect to Haar measure replaces the summation over the group elements which plays such a central role in the representation theory of finite groups. Of course every separable locally compact group will also have a left invariant Haar measure. However, it need not have a measure which is both left and right invariant. Groups G for which $M(G) < \infty$ have especially simple properties. They turn out to be precisely the compact groups. The hypothesis of separability is not important and is made chiefly for convenience. Most theorems generalize to the non-separable case, but their statements and proofs become somewhat more complicated.

We shall also confine our attention, from now on, to representations L, such that the space V is a separable complex Hilbert space (which we shall call $H(L)$), which have the following further properties.

(a) Each L_x is a *unitary operator* in the sense that $\| L_x(\phi) \| = \| \phi \|$ for all ϕ in $H(L)$ and $\phi \to L_x(\phi)$ has all of $H(L)$ for its range.

(b) For each ϕ in $H(L)$, the mapping $x \to L_x(\phi)$ is continuous from G to $H(L)$. We recall that a complex Hilbert space is a vector space H over the complex numbers equipped with an "inner product" $\phi, \psi \to (\phi, \psi)$ and that this inner product has the following properties: (i) $(\phi \cdot \psi) = \overline{(\psi \cdot \phi)}$; (ii) $(\phi \cdot \phi) > 0$ if $\phi \neq 0$; (iii) for each fixed ψ, $(\phi \cdot \psi)$ is linear as a function of ϕ; (iv) if we set $\| \phi \| = \sqrt{(\phi \cdot \phi)}$, then with respect to the "distance" $\| \phi - \psi \|$, H is a complete metric space. We remark that the continuity hypothesis labeled (b) above is implied by the superficially much weaker hypothesis (b′): for each ϕ and ψ in $H(L)$, $x \to (L_x(\phi) \cdot \psi)$ is a measurable function on G.

The notions of equivalence, direct sum, subrepresentation, etc., are defined in the present context much as in the first lecture. However, some changes are needed. We shall only call the representations L and M equivalent if there exists a unitary operator from $H(L)$ to all of $H(M)$ such that $WL_xW^{-1} = M_x$ for all x. We define the direct sum $L^1 \oplus L^2 \oplus \cdots$ of an infinite sequence $L^1, L^2, \cdots$ of representations as the representation whose space is the Hilbert space of all sequences $\phi_1, \phi_2, \cdots$ such that $\phi_j \in H(L^j)$ and $\sum_{j=1}^{\infty} \| \phi_j \|^2 < \infty$ and which is such that $(L^1 \oplus L^2 \cdots)_x(\phi_1, \phi_2, \cdots) = L^1_x(\phi_1), L^2_x(\phi_2), \cdots$. An invariant subspace of $H(L)$ will be a Hilbert space if and only if it is closed. Thus we associate subrepresentations only with closed invariant subspaces. Let H_1 be any closed invariant subspace of $H(L)$ and let $H_1^{\perp}$ be the set of all ψ in $H(L)$ such that $(\phi \cdot \psi) = 0$ for all ϕ in H_1. It is easy to show that $H_1^{\perp}$ is also invariant and that L is equivalent to the direct sum of $L^{H_1}\, L^{H_1^{\perp}}$.

We shall call a continuous linear transformation T from $H(L)$ to $H(M)$ an *intertwining operator* if $TL_x = M_xT$ for all x. If T is an intertwining operator, let N_T denote the set of all ϕ such that $T(\phi) = 0$, and let R_T denote the closure of the set of all $T(\psi)$. It is easy to verify that N_T and R_T are closed invariant subspaces of $H(L)$ and $H(M)$ respectively. Moreover, T restricted to N_T is an intertwining operator for L^{N_T} and M^{R_T}. Actually, it can be shown that the restriction of T to N_T is of the form UH where H is self-adjoint and U is unitary and U sets up an equivalence between L^{N_T} and M^{R_T}. The fact that L^{N_T} and M^{R_T} are equivalent is a modern version of a celebrated lemma of Schur. As a corollary we conclude that there exists a non-trivial intertwining operator for L and M if and only if some subrepresentation of L is equivalent to some subrepresentation of M. When no non-zero intertwining operator exists, we shall say that L and M are *disjoint*. It is easy to see that L and M are disjoint if and only if every invariant subspace of $H(L \oplus M)$ is of the form $H_1 \oplus H_2$ where $H_1 \subseteq H(L)$ and $H_2 \subseteq H(M)$. Thus, our terminology is consistent with that of the first lecture.

Let L be a representation and let $R(L)$ denote the set of all intertwining operators of L with L. $R(L)$ is clearly a subalgebra of the algebra of all bounded linear operators in $H(L)$. A closed subspace of $H(L)$ is completely described by the projection operator P where the range is this closed subspace. One sees easily that the subspace is invariant if and only if the projection is in $R(L)$. We shall denote the corresponding subrepresentation by L^P. It turns out that L^P is disjoint from L^{1-P} if and only if P is in the center of $R(L)$. As in the first lecture we shall say that an invariant subspace H_1 of $H(L)$ is central if L^{H_1} and $L^{H_1^{\perp}}$ are disjoint. It follows that H_1 is central if and only if the projection P on H_1 is in the center of $R(L)$ and we shall refer to the Boolean algebra formed by this family of projections as the center of L. When the center of L contains only 0 and 1, we shall say (again as in the first lecture) that L is *primary*. If the identity operator of $H(L)$

is a sum of minimal projections in the center of $R(L)$, that is, if the center of L is an atomic Boolean algebra, we shall say that L has a *discrete center*. Let this be the case and let $P^1, P^2 \cdots$ be the projections in question. Then the subrepresentations L^{P_j} will be central and primary and we have $L \simeq \sum_{j=1}^{\infty} L^{P_j}$. This decomposition is the *canonical primary decomposition*.

A primary representation will be said to be of type I if it has irreducible subrepresentations. The following statements are easy consequences of Schur's lemma. If L is primary and of type I, then all of its irreducible subrepresentations are equivalent and L is a direct sum of irreducible subrepresentations. If L and M are type I primary representations whose irreducible subrepresentations are inequivalent, then L and M are disjoint. One can also prove that $L \oplus L \oplus L \cdots$ (α terms) is equivalent to $L \oplus L \oplus L \cdots$ (β terms) if and only if $\alpha = \beta$. Thus a direct sum of type I primary representations is determined to within equivalence by assigning a "multiplicity" to each equivalence class of irreducible representations.

The *regular representation* R of a separable locally compact group G is the representation whose Hilbert space $H(R)$ is $L^2(G, \mu)$ where μ is right invariant Haar measure and whose operators R_x are the translation operators $f(y) \to f(yx)$.

As we shall see in the next lecture, the representation theory of separable locally compact groups differs in several important respects from the representation theory of finite groups. However, for compact groups the theory is almost exactly the same as for finite groups. They key fact is the famous Peter-Weyl theorem which we may formulate as follows: the regular representation of a compact group is a direct sum of finite dimensional irreducible representations and each occurs with a multiplicity equal to its dimensions. From this and relatively easy auxiliary arguments we can deduce the following: (1) every representation of a compact group has a discrete center; (2) every primary representation of a compact group is of type I; (3) every irreducible representation of a compact group is finite dimensional and is equivalent to a subrepresentation of the regular representation.

Since the irreducible representations of compact groups are all finite dimensional, they have characters and we may speak of the irreducible characters of a compact group. The canonical primary decomposition of a representation L of the compact group G may be constructed from the characters by formulae strictly analogous to those which apply to finite groups. If we set $f_\chi = \int_G L_x(f) \overline{\chi(x)}\, d\mu(x)$ where μ is the Haar measure in G such that $\mu(G) = 1$, then $f = \sum_{\chi \in \hat{G}} f_\chi$ where convergence is in the sense of the $H(L)$ norm and $\hat{G}$ is the set of all irreducible characters of G. In the special case in which G is commutative as well as compact and

L is the regular representation, f is a member of $L^2(G,\mu)$ and $f_\chi(y) = \int_G f(yx)\,\bar{\chi}(x)\,d\mu(x) = \chi(y)\,c(\chi)$ where $c(\chi) = \int_G f(x)\bar{\chi}(x)\,d\mu(x)$. Thus we have the dual formulae $f(y) = \sum_{\chi \in \hat{G}} c(\chi)\,\chi(y)$, $c(\chi) = \int f(x)\,\bar{\chi}(x)\,d\mu(x)$, generalizing those of the beginning of the lecture. When G is the additive group of all real numbers modulo the subgroup of all integral multiples of 2π, then $\hat{G}$ is the set of all functions of the form $x \to e^{inx}$ where n is an integer. The formulae reduce to those of the Fourier expansion theorem.

In the general compact commutative case, they take on a more symmetrical form if we notice that $\hat{G}$ is a discrete group whose Haar measure ν may be taken as the counting measure. Then $\sum_{\chi \in \hat{G}} c(\chi)\,\chi(y) = \int_{\hat{G}} c(\chi)\,\chi(y)\,d\nu(\chi)$ and we have $f(y) = \int_{\hat{G}} c(\chi)\,\chi(y)\,d\nu(\chi)$ and $c(\chi) = \int_G f(x)\,\bar{\chi}(x)\,d\mu(x)$. In this form the formulae are valid for all commutative locally compact groups whether they are compact or not. For each separable locally compact commutative group G the group $\hat{G}$ of all continuous irreducible characters is separable and locally compact in the so-called "compact open" topology, that is, the topology for which $\chi_n \to \chi$ if and only if $\chi_n(x) \to \chi(x)$ uniformly on compact subsets of G. The Haar measure ν in $\hat{G}$ is arbitrary up to a multiplicative positive constant, and it can be shown that there is just one way of choosing it so that $\int_{\hat{G}} |c(\chi)|^2 d\nu(\chi) = \int_G |f(x)|^2 d\mu(x)$ where f is any member of $L^2(G,\mu) \cap L^1(G,\mu)$ and $c(\chi) = \int f(x)\,\bar{\chi}(x)d\mu(x)$. More generally it can be shown that there is a unique unitary mapping $f \to \hat{f}$ of $L^2(G,\mu)$ onto $L^2(\hat{G},\nu)$ having the property that $\hat{f}(\chi) = c(\chi)$ whenever f is in $L^2(G,\mu)$ $\cap\, L^1(G,\mu)$. When $\hat{f} \in L^2(\hat{G},\nu) \cap L^1(\hat{G},\nu)$, f may be recovered from it by the formula

$$f(x) = \int \hat{f}(\chi)\,\chi(x)\,d\nu(\chi).$$

When G is the real line, $\hat{G}$ is also the real line and the mapping $f \to \hat{f}$ reduces to the classical Fourier transform. We have a rather natural common generalization of the theory of Fourier transforms in the theory of the mapping $f \to \hat{f}$ of $L^2(G,\mu)$ on $L^2(\hat{G},\nu)$. That this generalization exists seems to have been first observed by A. Weil.

LECTURE III

It follows from the considerations of the first two lectures that the problems of finding all continuous unitary representations of a compact group can be reduced to the problem of finding all of the irreducible representations of the group. Two questions immediately suggest themselves: (A) To what extent is the reduction in question possible when the group is locally compact but not compact. (B) For which groups and to what extent is it possible to find all irreducible representations. These are complicated questions on which research is still active, and it would take many lectures to treat them at all adequately. Nevertheless, we shall attempt a summary account which we hope will give the reader some idea of what is involved.

As far as the answer to question (A) is concerned, groups fall sharply into two categories depending upon whether or not every primary representation is of type I. Groups having only type I representations are said to be of type I. For such groups there is a subtle but quite complete and adequate decomposition theory in which sums are replaced by integrals. This theory is most easily described for commutative groups and may be regarded as a generalization of the Hahn-Hellinger theory which describes the unitary equivalence classes of self-adjoint operators. According to a well-known theorem of M. H. Stone, every continuous unitary representation of the additive group of the real line may be put into the form $t \to e^{itH}$ where H is self-adjoint. Moreover, the correspondence thus set up between equivalence classes of representations and equivalence classes of self-adjoint operators is one-to-one and onto. If this correspondence is used, the Hahn-Hellinger theory may be converted to one analyzing equivalence classes of unitary representations of the line. This converted theory has a more or less immediate generalization in which the real line is replaced by any separable locally compact commutative group. It may be described as follows. For each finite Borel measure μ in the dual $\hat{G}$ of the separable locally compact group G we may define a representation L^μ by taking $L^2(\hat{G},\mu)$ as $H(L^\mu)$ and setting $(L^\mu_x(f))(\chi) = \chi(x)f(\chi)$. Concerning the L^μ one can prove the following propositions: (I) L^μ and L^ν are equivalent if and only if μ and ν have the same sets of measure zero; (II) a representation L is of the form L^μ if and only if every subrepresentation is central; (III) L^μ and L^ν are disjoint if and only if there exist disjoint Borel sets E and F in $\hat{G}$ such that $E \cup F = \hat{G}$ and $\mu(E) = \nu(F) = 0$; (IV) if U is any continuous unitary representation of G, then there exist unique orthogonal central projections $P_\infty, P_1, P_2 \cdots$ in $R(L)$ such that $P_\infty + P_1 + P_2 + \cdots = 1$, and such that U^{P_j} is of the form jL^{μ_j}. In general we shall say that a representation is multiplicity-free if every subrepresentation is central. Thus, I, II, and III tell us how to reduce the study of multiplicity-free representations to the study of measure classes in $\hat{G}$ and IV tells us how to reduce the study of general representations to that of multiplicity-free representations.

When G is not commutative, $\hat{G}$ is just the set of all equivalence classes of irreducible representations of G and no longer has either a group structure or a well-behaved topology. On the other hand, one can define the notion of Borel set in $\hat{G}$ in a natural way and it turns out that groups fall naturally into two classes according to whether the Borel sets behave in a "regular" or "irregular" fashion. When they behave "regularly" $\hat{G}$ is said to be *smooth* and we can define a representation L^μ for each finite Borel measure μ in $\hat{G}$. The definition of L^μ is rather different from that given above in the commutative case but reduces to it when G is commutative. Under the hypothesis that G is of type I and $\hat{G}$ smooth, one can prove the propositions I, II, III and IV listed above and describe the general representation of G in terms of measure classes in $\hat{G}$. Since it has been shown by Glimm that $\hat{G}$ is smooth if and only if G is of type I, it follows that one has an analogue of the Hahn-Hellinger theory for all type I groups. In addition to the compact groups and the locally compact commutative groups, the type I groups include all semi-simple Lie groups (and hence in particular the "classical" matrix groups) and many special groups of importance in physics such as the crystallographic groups, the inhomomogeneous Lorentz group, and the Euclidean group.

Decomposition theory is in a much less satisfactory state for groups which have primary representations which are not of type I. This is partly because $\hat{G}$ fails to have "nice" properties in this case and partly because primary representations which are not of type I cannot be expressed in terms of irreducibles in any helpful fashion. While Mautner has applied von Neumann's direct integral theory to show that any representation may be decomposed as a direct integral of irreducibles, this decomposition is extremely non-unique when applied to non-type I primary representations. The situation is much too complicated to be discussed further here and we shall close our discussion of question (A) with a few remarks indicating the connection of the theory of non-type I primary representations with the von Neumann-Murray theory of operator algebras. If L is any representation, then the commuting algebra $R(L)$ defined in Lecture II is an example of what von Neumann and Murray called a *ring of operators*. Such a ring is called a *factor* if its center contains only multiples of the identity, and it follows at once that L is primary if and only if $R(L)$ is a factor. A factor is said to be of *type I* if it is isomorphic to the ring of all bounded operators on some Hilbert space. It can be shown that L is a type I primary representation if and only if $R(L)$ is a factor of type I. One of the fundamental contributions of von Neumann and Murray was to show that there exist factors which are not of type I and that they include those of the form $R(L)$ where L is a group representation.

We shall begin our discussion of question (B) by describing a general method for constructing unitary representations of groups out of unitary representations of their subgroups. Let H be any closed subgroup of the separable locally compact group G. Let us denote by G/H the set of all right cosets Hx. G/H is itself a separable locally compact topological space in a natural way and each $x \in G$ defines a homeomorphism $Hy \to Hyx$ of this space with itself. Suppose now that there exists a measure μ in G/H such that the measure of every Borel subset is invariant under the homeomorphisms $Hy \to Hyx$. Let L be any unitary representation of H. We shall show how to use μ and L to construct a unitary representation of G which we shall denote by U^L and call the *representation of G induced by L*. The space $H(U^L)$ of U^L is the set of all Borel functions f from G to $H(L)$ having the following two properties:

(a) $f(\xi x) = L_\xi f(x)$ for all $\xi \in H$, $x \in G$,

(b) $\int_{G/H} (f(x) \cdot f(x))\, d\mu(x) < \infty$.

Property (b) needs explanation since $f(x) \cdot f(x)$ is defined on G not on G/H. We note, however, that $(f(\xi x) \cdot f(\xi(x)) = (L_\xi f(x) \cdot L_\xi f(x))$ by (a) and hence, since L_ξ is unitary, we conclude that $(f(\xi x) \cdot f(\xi x)) = (f(x) . f(x))$. Thus $(f(x) \cdot f(x))$ is constant on the right H cosets and so defines a function on G/H. We make $H(U^L)$ into a Hilbert space by defining the inner product of f and g to be $\int_{G/H} (f(x) \cdot g(x)\, d\mu(x))$ and identifying functions which are almost everywhere equal. Now let us set $(U_x^L f)(y) = f(yx)$. We verify without difficulty that each U_x^L is a unitary operator in $H(U^L)$ and that $x \to U_x^L$ is a continuous unitary representation of G. Though this construction seems to depend upon the existence of the invariant measure μ, such is not in fact the case. Complicating the construction slightly, it is possible to get by with a measure which is only *quasi-invariant* in the sense that its sets of measure zero are invariant. It can be shown that quasi-invariant measures always exist and that changing from one quasi-invariant measure to another does not change the equivalence class of U^L. Thus U^L is uniquely defined by L when L is any unitary representation of any closed subgroup H of G.

Using the construction $L \to U^L$, it is possible to give an explicit description of all the irreducible representations of many interesting groups. Let the separable locally compact group G have a closed commutative normal subgroup N and a second closed subgroup K such that $N \cap K = e$, $NK = G$. Then every element x of G may be written uniquely in the form nk where $n \in N$ and $k \in K$ and we have $(n_1, k_1)(n_2, k_2) = n_1(k_1 n_2 k_1^{-1}), k_1 k_2$. We shall say that G is a *semi-direct product* of its subgroup N and K. For example, the group of all permutations of three objects is a semi-direct product of a normal subgroup of order three and a subgroup of order two. Moreover, the Euclidean group of all rigid motions in three-dimensional Euclidean space is a semi-direct product of the normal subgroup of all translations and the subgroup of all rotations about a fixed point. In the general case, we may proceed as follows to construct a large family of irreducible representations of G from the characters χ of N and the irreducible representations of certain subgroups of K. For each $\chi \in \hat{N}$, let K_χ denote the subgroup of K consisting of all elements with the property that $\chi(knk^{-1}) = \chi(n)$ for all $n \in N$. For each irreducible representation L of K_χ it is easy to see that $n, k \to \chi(n)L_k$ defines an irreducible representation χL of NK_χ. Moreover, though it is not obvious, one can prove that $U^{\chi L}$ is an irreducible representation of G. Concerning the equivalence of these irreducible representations one can say the following: (1) $U^{\chi L_1}$ is equivalent to $U^{\chi L_2}$ if and only if L_1 and L_2 are equivalent representations of K_χ; (2) if χ_1 and χ_2 "lie in the same K orbit," that is, if k exists so that $\chi_1(knk^{-1}) = \chi_2(n)$ for all $n \in N$, then every $U^{\chi_1 L_1}$ is equivalent to some $U^{\chi_2 L_2}$; (3) if χ_1 and χ_2 lie in different K orbits, then $U^{\chi_1 L}$ is not equivalent to any $U^{\chi_2 M}$. While it is not always true that every irreducible representation of G is equivalent to some $U^{\chi L}$, it is true in many important cases. In particular, it is true whenever there exists a Borel subset of $\hat{N}$ which meets each K orbit just once.

Consider the special case in which N is the group of all translations in Euclidean three-space and K is the group of all rotations about $(0,0,0)$. The members of $\hat{N}$ are the functions $x, y, z \to e^{i(\lambda x + \mu y + \nu z)}$ where λ, μ, ν varies over all triples of real numbers, and it is easy to see that λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 define characters in the same K orbit if and only if $\lambda_1^2 + \mu_1^2 + \nu_1^2 = \lambda_2^2 + \mu_2^2 + \nu_2^2$. Since the set of all $0, 0, \nu$ ($\nu \geqq 0$) is a Borel set, we see that every irreducible representation of G is of the form $U^{\chi_\nu L}$ where χ_v is the character $x, y, z \to e^{i\nu z}$ and L is an irreducible representation of K_{χ_ν}. Now if $\nu > 0$, K_{χ_ν} is isomorphic to the group of all rotations about a fixed axis and so is commutative. Thus its irreducible representations are one-dimensional and are defined by the characters $\theta \to e^{in\theta}$ where n is an integer. Let $V^{\nu,n}$ denote the irreducible representation $U^{\chi_\nu L^n}$, where $L_t^n = e^{int}I$. Then the $V^{\nu,n}$ are inequivalent infinite dimensional irreducible representations of G and include all irreducible representations of G except those of the form $U^{\chi_0 L}$. Now $K_{\chi_0} = K$ and $\chi_0(x, y, z) \equiv 1$. Thus the irreducible representations of the form $U^{\chi_0 L}$ are just those of the form $nk \to L_k$ where L varies over the (necessarily finite dimensional) irreducible representations of K. The irreducible representations of K are closely related to expansions in surface harmonics and may be described as follows. Let S be the surface of the unit sphere about the origin in three-space. For each integer n let P_n denote the ($2n + 1$ dimensional) vector space of all homogeneous nth degree polynomials which satisfy Laplace's equation. Then the restrictions to S of the functions in P_n form a $2n + 1$ dimensional subspace of $L^2(S, \nu)$ where ν is the area measure in S. This subspace is invariant under rotation and defines an irreducible representation of the rotation group K. It can be shown that every irreducible representation of K is equivalent to one of these.

The analysis we have given of the representations of semi-direct products can be extended to a theory relating the irreducible representations of G to those of a normal subgroup N and certain of the subgroups of G/N. To the extent that this theory is complete, one is reduced (by induction) to studying the irreducible representations of simple groups. When the simple group is compact and connected, one has a complete theory available. This theory was worked out by E. Cartan and H. Weyl in the 1920's. For other simple groups much less is known. Various finite simple groups have been completely studied, the alternating groups for example, and certain linear groups over finite fields. However, there are still many finite simple groups whose irreducible representations are not known. In the past sixteen years the non-compact semi-simple Lie groups have been assiduously studied by a number of mathematicians, especially Gelfand, Naimark, and Graev in Russia, and Harish-Chandra in this country. However, complete results are available only in rather special cases. So far all of the irreducible representations one has found of these groups have been constructed by the inducing process and certain mild variants of it.